Yiddish of the Day

dirt = shmutz = שמוץ

to chat = shmooz = שמועס

to sweat = shvitz = שוויץ

Tensor-Products Cont

Last time: For vector spaces $V_1 \ldots V_n$, W over $\mathbb{F}$, we saw that the "$\otimes$" operation gave us the bijection

$$\mathrm{Mult}(V_1 \times \cdots \times V_n, W) \cong \mathcal{L}(\underline{V_1 \otimes \cdots \otimes V_n}, W)$$

$$f: V_1 \times \cdots \times V_n \to W \longmapsto \tilde{f}: V_1 \otimes \cdots \otimes V_n \to W$$

Moreover, we mentioned

- how we can find a basis for it
- how it "behaves" with linear maps

Let us now compute some examples. Here is a quick comment to note

- The thm about the basis of $V \otimes W$ can be used to show certain examples are isomorphisms just by counting dimensions. However! Sometimes the form of the isomorphism is more important (ie, what is the actual map)

ex) Prove $\mathbb{F} \otimes \mathbb{F} \cong \mathbb{F}$

Pf 1) Dimension

$\dim(\mathbb{F} \otimes \mathbb{F}) = \dim(\mathbb{F})$

✓

Pf 2) Universal property

Define $T: \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ to be $F(a,b) = ab$. Thus we get ! linear map $\mathbb{F} \otimes \mathbb{F} \to \mathbb{F}$ that sends

$\tilde{J}(a \otimes b) = ab$. This is injective and hence an isomorphism

(Rmk: Compare the above to $\mathbb{F} \oplus \mathbb{F}$)

ii) Prove that $\exists$ iso $V \otimes W \cong W \otimes V$ that sends $v \otimes w \to w \otimes v$

Pf) Define the map $V \times W \xrightarrow{f} W \otimes V$

$(v, w) \rightsquigarrow w \otimes v$

Note this is bilinear: Indeed

$f(v_1 + v_2, w) = w \otimes (v_1 + v_2) = w \otimes v_1 + w \otimes v_2 = f(v_1, w) + f(v_2, w)$

Similarly $f(v, w_1 + w_2) = f(v, w_1) + f(v, w_2)$

$r f(v, w) = r(w \otimes v) = w \otimes rv = f(rv, w)$

$= rw \otimes v$

$= f(v, rw)$

$\rightsquigarrow$ We get unique linear map

$\tilde{f} : V \otimes W \to W \otimes V$ that sends

$\tilde{f}(v \otimes w) = w \otimes v$

<u>Construct inverse</u>: Define $g : W \times V \to V \otimes W$

$g(w, v) = v \otimes w$. Again this is bilinear, so

we get linear map that sends

$\tilde{g}(w \otimes v) = v \otimes w$

<u>Check</u>: $\tilde{f}$ and $\tilde{g}$ are inverses of each other $\square$

$\Rightarrow \tilde{f} : V \otimes W \to W \otimes V$ is an isomorphism.

iii) Let V_1, V_2, W be vector spaces and $\mathbb{F}$. Prove $\exists$ iso

$$(V_1 \oplus V_2) \otimes W \cong V_1 \otimes W \oplus V_2 \otimes W$$

that sends the simple tensor $(v_1, v_2) \otimes w \to (v_1 \otimes w, v_2 \otimes w)$

Pf) Define $f: (V_1 \oplus V_2) \times W \to V_1 \otimes W \oplus V_2 \otimes W$

$$f((v_1, v_2), w) = (v_1 \otimes w, v_2 \otimes w)$$

<u>Check</u>: this f is bilinear.

- $f(((v_1, v_2) + (v_1', v_2')), w) \overset{\text{check}}{=} f((v_1, v_2), w) + f((v_1', v_2'), w)$

$\rightsquigarrow$ get linear map $\tilde{f}: (V_1 \oplus V_2) \otimes W \to V_1 \otimes W \oplus V_2 \otimes W$

To define an inverse from the direct sum we will define two maps out of $V_1 \otimes W$ and $V_2 \otimes W$

$$g_1 : V_1 \times W \longrightarrow (V_1 \oplus V_2) \otimes W$$
$$(v_1, w) \longmapsto (v_1, 0) \otimes w$$

$$g_2 : V_2 \times W \longrightarrow (V_1 \oplus V_2) \otimes W$$
$$(v_2, w) \longrightarrow (0, v_2) \otimes w$$

$\leadsto$ g_1 and g_2 are bilinear so we get 2 linear maps

$$\tilde{g}_1 : V_1 \otimes W \longrightarrow (V_1 \oplus V_2) \otimes W$$
$$\tilde{g}_1(v_1 \otimes w) = (v_1, 0) \otimes w$$

$$\tilde{g}_2 : V_2 \otimes W \longrightarrow (V_1 \oplus V_2) \otimes W$$
$$\tilde{g}_2(v_2 \otimes w) = (0, v_2) \otimes w$$

Then we get a map

$$\tilde{g} : (V_1 \otimes W) \oplus (V_2 \otimes W) \longrightarrow (V_1 \oplus V_2) \otimes W$$

$$\tilde{g} = \tilde{g}_1 + \tilde{g}_2$$

$$\tilde{g}\big((v_1 \otimes w_1), (v_2 \otimes w_2)\big) = \tilde{g}_1(v_1 \otimes w_1) + \tilde{g}_2(v_2 \otimes w_2)$$
$$= (v_1, 0) \otimes w_1 + (0, v_2) \otimes w_2$$

Then we can show that

check ☺

$$\left\{ \begin{array}{l} \tilde{g} \circ \tilde{f}\big((v_1, v_2) \otimes w\big) = (v_1, v_2) \otimes w \\ \tilde{f} \circ \tilde{g}\big((v_1 \otimes w_1), (v_2 \otimes w_2)\big) = \big((v_1 \otimes w_1), (v_2 \otimes w_2)\big) \end{array} \right.$$

$\Rightarrow \tilde{f}$ is an isomorphism $\square$

ex) $\mathbb{R}^3 \otimes \mathbb{R}^2 \cong \mathbb{R}^6$

$$(\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}) \otimes (\mathbb{R}^2) \cong (\mathbb{R}\otimes\mathbb{R}^2)\oplus(\mathbb{R}\otimes\mathbb{R}^2)\oplus(\mathbb{R}\otimes\mathbb{R}^2)$$

$$\cong (\mathbb{R}\otimes(\mathbb{R}\oplus\mathbb{R}))\oplus(\mathbb{R}\otimes(\mathbb{R}\oplus\mathbb{R}))\oplus(\mathbb{R}\otimes(\mathbb{R}\oplus\mathbb{R}))$$

$$\cong \mathbb{R}\otimes\mathbb{R}\oplus\mathbb{R}\otimes\mathbb{R}\oplus\mathbb{R}\otimes\mathbb{R}\oplus\mathbb{R}\otimes\mathbb{R}\oplus\mathbb{R}\otimes\mathbb{R}\oplus\mathbb{R}\otimes\mathbb{R}$$

$$\cong \mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R} = \mathbb{R}^6$$

iv) For V_1, V_2, W $\exists!$ isomorphism

$$(V_1 \otimes V_2) \otimes W \longrightarrow V_1 \otimes (V_2 \otimes W)$$

that sends the simple tensor $(v_1 \otimes v_2) \otimes w \longmapsto v_1 \otimes (v_2 \otimes w)$

(ie, the $\otimes$-product is associative)

Pf) Fix $w \in W$, and define the map

$$f_w : V_1 \times V_2 \longrightarrow V_1 \otimes (V_2 \otimes W)$$

by $(v_1, v_2) \longmapsto v_1 \otimes (v_2 \otimes w)$. Then this is bilinear so this gives us a well defined linear map

$$\tilde{f}_w : V_1 \otimes V_2 \longrightarrow V_1 \otimes (V_2 \otimes W)$$

$$\tilde{f}(v_1 \otimes v_2) = v_1 \otimes (v_2 \otimes w)$$

Now define the map

$$g: V_1 \otimes V_2 \times W \longrightarrow V_1 \otimes (V_2 \otimes W)$$

$$(v_1 \otimes v_2 \,,\, w) \longmapsto \tilde{f}_w(v_1 \otimes v_2) = v_1 \otimes (v_2 \otimes w)$$

Construct the inverse map!

Claim: g is bilinear. This follows because $\tilde{f}_w$ is linear

$\Rightarrow$ have linear map $\tilde{g}: (V_1 \otimes V_2) \otimes W \longrightarrow V_1 \otimes (V_2 \otimes W)$
sending $\tilde{g}((v_1 \otimes v_2) \otimes w) = v_1 \otimes (v_2 \otimes w)$.

v) There is a well-defined linear isomorphism $\mathbb{R}^n \otimes \mathbb{R}^n \longrightarrow M_n(\mathbb{R})$
sending a simple tensor

("outer-product") $\vec{v} \otimes \vec{w} \longmapsto \vec{v}\,\vec{w}^{tr}$

Pf) Define the map $f: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow M_n(\mathbb{R})$

$(\vec{v}, \vec{w}) \mapsto \vec{v}\vec{w}^{tr}$. This is bilinear so get linear map $\tilde{f} : \mathbb{R}^n \otimes \mathbb{R}^n \to M_n(\mathbb{R})$

$(e_i \otimes e_j) \mapsto e_i e_j^{tr}$ = matrix with 1 in ij spot, zero everywhere else.

Sends basis to basis hence iso.

Rmk: There is a sense in which row vectors $(x_1, \ldots x_n) \in (\mathbb{R}^n)^*$ Indeed choose a basis $\mathcal{B} = (v_1 \ldots v_n)$ for V and let $\varphi \in V^*$. Then $[\ ]_\mathcal{B}$ is just a $\underline{\quad \times \quad}$ matrix (i.e. $\ldots$)

So column vectors = "elements of $\mathbb{R}^n$"
row vectors = "elements of $(\mathbb{R}^n)^*$"

With this, and the last prop in mind, consider the following

Prop: $V \otimes V^* \cong \mathcal{L}(V,V)$ $\quad$ ($|x\rangle\langle x|$)

(Rmk: This is how tensors often show up in physics!)

Pf) Define map $V \times V^* \to \mathcal{L}(V,V)$

$$(v, g) \longmapsto T: V \to V$$
$$T_v(w) := g(w)v \qquad \left(T_{v,g}(w) = g(w)v \right)$$

This defines a function

$$V \times V^* \to \mathcal{L}(V,V)$$

This is in-fact a bilinear map!

Since $g(w)(v_1+v_2) = g(w)v_1 + g(w)v_2$

$$= T_{v_1}(w) + T_{v_2}(w)$$

Also $(g_1+g_2)(w)v = (g_1(w)+g_2(w))v$

$$= g_1(w)v + g_2(w)v$$

$\rightsquigarrow$ get a linear map

$$V \otimes V^* \longrightarrow \mathcal{L}(V,V)$$

Let $T: V \to V$ be an element of $\mathcal{L}(V,V)$. Write T as a matrix (wrt any basis) $[T] = (a_{ij})$

(that is $T(e_k) = \sum_i a_{ik} e_i$) (A)

Then consider the tensor

(this gives us a map $\mathcal{L}(V,V) \to V \otimes V^*$) $\sum_{i,j} a_{ij}\, e_i \otimes e_j^* \in V \otimes V^*$

(these a_{ij} are the same as the components of (a_{ij}))

ex: If $\dim V = 2$ with $[T] = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$\leadsto a_{11} e_1 \otimes e_1^* + a_{12} e_1 \otimes e_2^* + a_{21} e_2 \otimes e_1^* + a_{22} e_2 \otimes e_2^*$

Now $\underbrace{\left(\sum_{i,j} a_{ij}\, e_i \otimes e_j^*\right)}_{\text{viewing as function now}}(e_k) = \sum_i a_{ik} e_i = T(e_k)$ by ($\star$)

Thus
$$\mathcal{L}(V,V) \longrightarrow V \otimes V^* \longrightarrow \mathcal{L}(V,V)$$
$$T \longmapsto \sum a_{ij} e_i \otimes e_j^* \longrightarrow \left(\sum a_{ij} e_i \otimes e_j^*\right)(-)$$
is the identity $\boxed{\smile}$

We denote this map we get from the bijection

$(v \otimes g)(v') = g(v')v$

HW

iv) Generalize the above to show

$$V \otimes W^* \simeq \mathcal{L}(V, W)$$

v) Use (iv) to prove

$$\mathcal{L}(V \otimes W, \mathbb{F}) \simeq \mathcal{L}(V, \mathcal{L}(W, \mathbb{F}))$$

(i.e. $(V \otimes W)^* \simeq \mathcal{L}(V, W^*)$)

"Unimportant" Rmk: Such an isomorphism as in (v) true more generally. Challenge Problem!

Prove that for V, W, Z vector spaces

$$\mathcal{L}(V \otimes W, Z) \simeq \mathcal{L}(V, \mathcal{L}(W, Z))$$

(the famous "$\otimes$-hom adjunction")

ex) $V = \mathbb{R}^2$ with standard basis (e_1, e_2).

Let's see what map $\mathbb{R}^2 \to \mathbb{R}^2$ (ie a 2×2 matrix) corresponds to the tensors described below

Let $g_1 \begin{pmatrix} x \\ y \end{pmatrix} = x$

1) $e_1 \otimes g$

- $(e_1 \otimes g)(e_1) = g(e_1) e_1 = e_1$ $\longrightarrow$ $e_1 \otimes g = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
- $(e_1 \otimes g)(e_2) = 0$

2) $e_2 \otimes g$

- $(e_2 \otimes g)(e_1) = g(e_1) e_2 = e_2$ $\qquad e_2 \otimes g = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
- $(e_2 \otimes g)(e_2) = 0$

For $g\begin{pmatrix}x\\y\end{pmatrix} = x+y$

1) $3e_1 \otimes g$

- $(3e_1 \otimes g)(e_1) = g(e_1)(3e_1) = 3e_1$
- $(3e_1 \otimes g)(e_2) = g(e_2)(3e_1) = 3e_1$

$$3e_1 \otimes g = \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix}$$

2) $-2e_2 \otimes g_2$

- $(-2e_2 \otimes g)(e_1) = -2e_2$
- $(-2e_1 \otimes g)(e_2) = -2e_2$

$$-2e_2 \otimes g = \begin{pmatrix} 0 & 0 \\ -2 & -2 \end{pmatrix}$$

3) Find $14e_1 \otimes (g_1 - g_2)$